数理解析研究所講究録
1140 巻 2000 年 52-60

A multiplication on the twisted tensor product

栗林 勝彦 (Katsuhiko Kuribayashi)
Okayama University of Science
三村 護 (Mamoru Mimura)
Okayama University
手塚 康誠 (Michishige Tezuka)
Ryukyu University

1 Introduction

Let G be a connected topological group. We define the right adjoint action $ad : G\times G\to G$ by $ad(g, h)=h^{-1}gh$. Then the cohomology $H^{*}(G;\mathbf{Z}/l)$ is regarded as a right $H^{*}(G;\mathbf{Z}/l)$-comodule under the coaction induced by the adjoint action. The comodule is denoted by $H^{*}(G;\mathbf{Z}/l)_{c}$ below. In this note, the algebra structure of

$$E:=\mathrm{Cotor}_{H^{*}(G;\mathbf{Z}/l)}(H^{*}(G;\mathbf{Z}/l)_{c}, \mathbf{Z}/l)$$

is considered from the viewpoint of the a differential graded algebra structure of the twisted tensor product due to Brown [1]. The existence of the following three spectral sequences motivates the consideration of the algebra structure of E.

(1) Let $G(\mathbf{F}_{q})$ be a finite Chevalley group of Lie type over the finite field $\mathbf{F}_{q}$ of q elements and l a prime number. By applying the Deligne spectral sequence in the case where the characteristic of $\mathbf{F}_{q}$ is prime to l, Tezuka [7] has constructed a spectral sequence converging to $H^{*}(BG(\mathbf{F}_{q});\mathbf{Z}/l)$. In particular if $q-1\equiv 0$ modulo l, then the E_{2}-term of the spectral sequence is isomorphic to E as an algebra for many cases.

(2) Let BLG be the classifying space of the loop group LG consisting of all continuous maps from the circle to G. Then there exists the

Eilenberg-Moore spectral sequence, whose E_2-term is isomorphic to E as an algebra, converging to $H^*(BLG; \mathbf{Z}/l)$.

(3) Let X be a simply connected finite CW-complex. Following Milnor's description of universal bundles over a space, we can regard the loop space ΩX, which is the subspace of the free loop space LX consisting of based loops, as a topological group G. Therefore we have the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/l)$ with $E_2 \cong E$ as an algebra.

One will know that it is important to clarify the algebra structure of E as the first step in computing those spectral sequences.

Let G be a connected complex Lie group with the same Lie type as that of a finite Chevalley group $G(\mathbf{F}_q)$. As for the cohomology algebras of $BG(\mathbf{F}_q)$ and BLG, Tezuka [15] has proposed a problem whether the cohomologies $H^*(BG(\mathbf{F}_q); \mathbf{Z}/l)$ and $H^*(BLG; \mathbf{Z}/l)$ are isomorphic as an algebra in the case where l is odd and divides $q-1$ but does not divide q or $l = 2$ and 4 divides $q-1$. As mentioned in [15], the answer is affirmative if the integral cohomology of G has no l-torsion. The main theorem in [6] and the explicit calculation of $H^*(BG(\mathbf{F}_q); \mathbf{Z}/l)$ due to Kleinerman [3] guarantee the result. To shed light on left part of the problem, we will consider the structure of E for the case where $H^*(G; \mathbf{Z})$ has l-torsion.

2 Results

Before stating our results, we recall a construction of the twisted tensor product due to Brown (see [1], [14] or [4]). Let A be a coalgebra over $\mathbf{Z}/l$ with coproduct ϕ_A and augmentation ε. Let L be a $\mathbf{Z}/lp$-subspace of A, $\iota : L \to A$ the inclusion and $\theta : A \to L$ a map such that $\theta \circ \iota = id_L$. We define the map $\bar{\theta} : A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota} : sL \to A$ by $\bar{\iota} = \iota \circ s^{-1}$, where $s : L \to sL$ is a suspension. Construct the tensor product $X = T(sL)$ and denote by ψ the product in $T(sL)$. The map $\bar{\theta}$ induces a map $A \to T(sL)$ which is again denoted by $\bar{\theta}$. Let I be the ideal of $T(sL)$ generated by $(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)$ (ker $\bar{\theta}$). The twisted tensor product (W, d) with respect to $\bar{\theta}$ is defined as follows; we put

$W = A \otimes X/I = A \otimes \bar{X}$ and define the differential operator d_W by

$$d_W = 1 \otimes d_{\bar{X}} + (1 \otimes \psi) \circ (1 \otimes \bar{\theta} \otimes 1) \circ (\phi_A \otimes 1), \quad \text{where}$$
$$d_{\bar{X}} = -\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A \circ \bar{\iota}.$$

We may denote the twisted tensor product W with respect to $\bar{\theta} : A \to sL$ by $A \otimes_\theta \bar{X}$.

Let G be a compact, simply connected, simple exceptional Lie group. Then it is known [9] that a suitable choice of a subspace L of $H^*(G; \mathbf{Z}/l)$ makes the twisted tensor product into an injective resolution $0 \to \mathbf{Z}/l \to H^*(G; \mathbf{Z}/l) \otimes_\theta \bar{X}$ over the coalgebra A. Moreover the algebra structure of $\bar{X}$ induces that of the complex

$$(\mathbf{Z}/l \square_{H^*(G;\mathbf{Z}/l)} (H^*(G; \mathbf{Z}/l) \otimes_\theta \bar{X}), 1 \square d_W) \cong (\bar{X}, d_{\bar{X}})$$

Consequently we have

$$\mathrm{Cotor}_{H^*(G;\mathbf{Z}/l)}(\mathbf{Z}/l, \mathbf{Z}/l) \cong H(\bar{X}, d_{\bar{X}}) \quad \text{as an algebra.}$$

In this note, we consider a multiplication m_W on the twisted tensor product $A \otimes_\theta \bar{X}$ for a Hopf algebra A, in the sense of Milnor and Moore [8], such that the differential d_W is derivative under the multiplication. In order to define a multiplication m_W explicitly, we will assume that the $\mathbf{Z}/l$-subspace L of A satisfies the following condition.

(I) *There exist the set Q of indecomposable elements of A and a basis $\{x_i\}$ of L such that $\{x_i\} \subset Q \cup Q^2$, where $Q^2 = \{\alpha^2 | \alpha \in Q \cap \mathit{Prim}\, A\}$ and, as an algebra,*

$$A \cong \bigotimes_{x_s \in S} \mathbf{Z}/p[x_s]/(x_s^{p^{n_s}}) \otimes \bigotimes_{x_t \in T} \Lambda(x_j) \ ,$$

where $S \cup T = Q \cap \{x_i\}$ and $S \cap T = \phi$. Moreover, we also assume that

(II) $(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)(\ker \bar{\theta}) = \mathbf{Z}/l\{(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)(x_i x_j) | x_i, x_j \in \{x_i\}, i \neq j\}$,

(III) *for any $a \in Q$, $\bar{\theta}(y a_i'') = 0$ for any $y \in \bar{A}$, where $\phi_A(a) = \sum_i a_i' \otimes a_i'' + a \otimes 1 + 1 \otimes a$* and that

(IV) *for any x and $y \in \{x_i\}$, $\bar{\theta}(xy) \neq 0$ if and only if $x = y$ and*

$x^2 \in Q^2$.

We mention here that the conditions (I), (II) (III) and (IV) hold in the cases $(PU(3),3)$, $(F_4,3)$, $(E_8,3)$, (E_6,p), (E_7,p) for $l = 2$ and 3 which have been studied by Kono, Mimura, Sambe and Shimada ([4],[5], [10], [11]).

The following is one of the our main theorem.

Theorem 2.1. *Let A be a Hopf algebra over $\mathbf{Z}/l$. For any elements $a \otimes \theta x$ and $b \otimes \theta y$ of $A \otimes_\theta \bar{X}$, define $m_W : A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X} \to A \otimes_\theta \bar{X}$ by*

$$m_W(a \otimes \theta x \otimes b \otimes \theta y) = a \otimes \theta x \cdot b \otimes \theta y = \sum_i (-1)^{|\theta x||b_i'|} ab_i' \otimes \theta(xb_i'')\theta y,$$

and

$$(\theta x_1 \cdots \theta x_s) \cdot a = (\theta x_1(\theta x_2(\cdots(\theta x_s \cdot a))\cdots),$$

where $\phi_A(b) = \sum_i b_i' \otimes b_i''$. If m_W is well-defined, then $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a differential graded algebra.

By comparing the differential algebra structure of the cobar resolution [13, 7.A, 1.2] of the left A-comodule $\mathbf{Z}/l$ and that of the twisted tensor product mentioned above, we can prove Theorem 1.

Theorem 2.2. *If $l = 2$ or 3 and the condition (I), (II), (III) and (IV) hold, then the multiplication m_W is well-defined.*

In the case where $A = H^*(E_8; \mathbf{Z}/5)$, explicit calculation for the differential d_W and the multiplication m_W on $A \otimes_\theta \bar{X}$ allow us to obtain the following theorem.

Theorem 2.3. *Let $A \otimes_\theta \bar{X}$ be the twisted tensor product of $H^*(E_8; \mathbf{Z}/5)$ constructed in [12]. Then $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a well-defined differential graded algebra.*

In the case where $A = H^*(E_8; \mathbf{Z}/2)$, indecomposable elements x on A can be chosen so that $\bar{\Delta}(x)$ is in $P \otimes P$, where P is the $\mathbf{Z}/2$-subspace of A consisting of primitive elements. Thanks to this fact, we can easily verify that the multiplication m_W is well-defined.

Theorem 2.4. *Let $A\otimes_\theta\bar{X}$ be the twisted tensor product of $H^*(E_8;\mathbf{Z}/2)$ constructed in [9]. Then $(A\otimes_\theta\bar{X}, d_W, m_W)$ is a well-defined differential graded algebra.*

In order to prove that the multiplication m_W induces the algebra structure on $\mathrm{Cotor}_A(A,\mathbf{Z}/p)$, it suffices to prove

Proposition 2.5. *Let p be a prime number and $\mu : A\otimes A\to A$ the multiplication of A. Then the map $m_W : A\otimes_\theta\bar{X}\otimes A\otimes_\theta\bar{X}\to A\otimes_\theta\bar{X}$ is a μ-morphism if m_W is well-defined, that is, the following diagram is commutative:*

$$\begin{tikzcd}
A\otimes_\theta\bar{X}\otimes A\otimes_\theta\bar{X} \arrow[r, "\psi_1"] \arrow[d, "m_W"'] & (A\otimes A)\otimes A\otimes_\theta\bar{X}\otimes A\otimes_\theta\bar{X} \arrow[d, "\mu\otimes m_W"] \\
A\otimes_\theta\bar{X} \arrow[r, "\psi_2"'] & A\otimes A\otimes_\theta\bar{X},
\end{tikzcd}$$

where ψ_1 and ψ_2 are the comodule structures of $A\otimes_\theta\bar{X}\otimes A\otimes_\theta\bar{X}$ and $A\otimes_\theta\bar{X}$ respectively.

Let A denote the mod l cohomology $H^*(G;\mathbf{Z}/p)$. Since $ad^*\otimes 1 : A\otimes\bar{X}\to A\square_A(A\otimes\bar{X})$ is the isomorphism with the inverse $1\otimes\varepsilon\otimes 1$, we can define a differential on $A\otimes\bar{X}$ by the compositions

$$A\otimes\bar{X}\xrightarrow{ad^*\otimes 1}A\square_A(A\otimes\bar{X})\xrightarrow{inc}A\otimes(A\otimes\bar{X})\xrightarrow{1\otimes d_W}A\otimes(A\otimes\bar{X})\xrightarrow{1\otimes\varepsilon\otimes 1}A\otimes\bar{X}.$$

A straightforward calculation for the differential $d : A\otimes\bar{X}\to A\otimes\bar{X}$ enables us to obtain the following explicit formula for d.

Lemma 2.6. *We write as $\Delta_A(x) = x\otimes 1 + 1\otimes x + \sum_i x_i'\otimes x_i''$ for $x\in A$. If x_i' is primitive for any i, then*

$$dx = -\sum_i(-1)^{|x_i''|(|x_i'|+1)}x_i''\otimes\theta x_i' + \sum_i(-1)^{|x_i'|}x_i'\otimes\theta x_i''\ .$$

The multiplication m_W on the twisted tensor product $A\otimes_\theta\bar{X}$ induces a multiplication m on $A\otimes\bar{X}$ defined by

$$\begin{gathered}
A\otimes\bar{X}\otimes A\otimes\bar{X}\xrightarrow{ad^*\otimes 1\otimes ad^*\otimes 1}A\square_A(A\otimes\bar{X})\otimes A\square_A(A\otimes\bar{X})\xrightarrow{inc} \\
A\otimes(A\otimes\bar{X})\otimes A\otimes(A\otimes\bar{X})\longrightarrow A\otimes A\otimes(A\otimes\bar{X})\otimes(A\otimes\bar{X})\xrightarrow{m_A\otimes m_W} \\
A\otimes(A\otimes\bar{X})\xrightarrow{1\otimes\varepsilon\otimes 1}A\otimes\bar{X}.
\end{gathered}$$

We can obtain an explicit formula for the multiplication m on $A \otimes \bar{X}$.

Lemma 2.7. *We write as* $\Delta_A(a) = a \otimes 1 + 1 \otimes a + \Sigma_i a'_i \otimes a''_i$ *for* $a \in A$. *If* a'_i *is primitive for any* i, *then*

$$\begin{aligned} \theta x \cdot a &= (-1)^{|\theta x||a|} a \otimes \theta x - \Sigma_i (-1)^{|a''_i||a'_i| + |a''_i||\theta x|} a''_i \otimes \theta(x a'_i) \\ &\quad + \Sigma_i (-1)^{|a'_i||\theta x|} a'_i \otimes \theta(x a''_i) \ . \end{aligned}$$

Thus we can obtain a differential graded algebra $(A \otimes \bar{X}, d, m)$. From the construction of this differential graded algebra, we have

Theorem 2.8. *For the case where* $A = H^*(G; \mathbf{Z}/l)$, *if the twisted tensor product* $(A \otimes_\theta \bar{X}, d_W, m_W)$ *is a well-defined differential graded algebra, then, as an algebra,*

$$\mathrm{Cotor}_{H^*(G;\mathbf{Z}/l)}(H^*(G; \mathbf{Z}/l)_c, \mathbf{Z}/l) \cong H(A \otimes \bar{X}, d, m).$$

The proofs of theorems and propositions in this note will be given in a further article [7].

This note will be concluded with some examples of the differential graded algebras $A \square_A (A \otimes_\theta \bar{X})$ for computing the algebras $\mathrm{Cotor}_A(A, \mathbf{Z}/l)$.

The case $(G, p) = (PU(3), 3)$.

$W' = A \square_A (A \otimes_\theta \bar{X}) = \mathbf{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1, x_3) \otimes \mathbf{Z}/3\{a_2, a_3, c_5, b_4\}/I,$

$$\begin{aligned} db_4 = -a_2 a_3,\ dc_5 &= a_3^2, \\ d(x_3) = x_2 \otimes a_2 &+ x_1 \otimes a_3, \end{aligned}$$

$$a_3 \cdot x_3 = -x_3 \otimes a_3 + x_1 \otimes c_5.$$

Therefore,we have, as a $\mathrm{Cotor}_{H^*(PU(3);\mathbf{Z}/3)}(\mathbf{Z}/3, \mathbf{Z}/3)$-module,

$$\begin{gathered} \mathrm{Cotor}_{H^*(PU(3);\mathbf{Z}/3)}(H^*(PU(3); \mathbf{Z}/3), \mathbf{Z}/3) \cong \\ \{\mathbf{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1) \otimes \mathbf{Z}/3[y_2, y_3, y_7, y_8, y_{12}]/(y_2 y_3, y_3^2, y_2 y_7, y_7^2, \\ y_2 y_8 + y_3 y_7) \\ \oplus x_3 \cdot (x_1 x_2^2, x_1 y_7, x_1 y_8 + x_2 y_7, x_2^2 y_2, y_3)\} \big/ (x_2 y_2 + x_1 y_3). \end{gathered}$$

The case $(G,p)=(F_4,3)$.

$$W'=A\Box_A(A\otimes_\theta \bar{X})=\mathbf{Z}/3[x_8]/(x_8^3)\otimes\Lambda(x_3,x_7,x_{11},x_{15})$$
$$\otimes\mathbf{Z}/3\{a_4,a_8,a_9,b_{12},b_{16},c_{17}\}/I,$$

$$d(x_j)=x_8\otimes a_{j-8+1}+x_{j-8}\otimes a_9 \quad (j=11,15),$$
$$d|_{\mathbf{Z}/3\{\ \}/I}=\text{the ordinary differential on } \mathbf{Z}/3\{\ \}/I,$$

$$a_9\cdot x_j=-x_j\otimes a_9+x_{j-8}\otimes c_{17} \quad (j=11,15).$$

The case $(G,p)=(E_6,3)$.

$$W'=A\Box_A(A\otimes_\theta \bar{X})=$$
$$\mathbf{Z}/3[x_8]/(x_8^3)\otimes\Lambda(x_3,x_7,x_9,x_{11},x_{15},x_{17})\otimes$$
$$\mathbf{Z}/3\{a_4,a_8,a_9,a_{10},b_{12},b_{16},b_{18},c_{17}\}/I,$$

$$d(x_j)=x_8\otimes a_{j-8+1}+x_{j-8}\otimes a_9 \quad (j=11,15,17),$$
$$d|_{\mathbf{Z}/3\{\ \}/I}=\text{the ordinary differential on } \mathbf{Z}/3\{\ \}/I,$$

$$a_9\cdot x_j=-x_j\otimes a_9+x_{j-8}\otimes c_{17} \quad (j=11,15,17).$$

The case $(E_7,3)$.

$$W'=A\Box_A(A\otimes_\theta \bar{X})=\mathbf{Z}/(x_8^3)\otimes\Lambda(x_3,x_7,x_{11},x_{15},x_{19},x_{27},x_{35})$$
$$\otimes\mathbf{Z}/3\{a_4,a_8,a_9,a_{20},b_{12},b_{16},b_{28},c_{17},e_{36}\}/I,$$

$$d(x_j)=x_8\otimes a_{j-8+1}+x_{j-8}\otimes a_9 \quad (j=11,15,27),$$
$$d(x_{35})=x_8\otimes b_{28}+x_{27}\otimes a_9-x_8^2\otimes a_{20}+x_{19}\otimes c_{17},$$
$$d|_{\mathbf{Z}/3\{\ \}/I}=\text{the ordinary differential on } \mathbf{Z}/3\{\ \}/I,$$

$$a_9\cdot x_j=-x_j\otimes a_9+x_{j-8}\otimes c_{17} \quad (j=11,15,27,35).$$

The case $(E_8,3)$.

$$W'=A\Box_A(A\otimes_\theta \bar{X})=$$
$$\mathbf{Z}/3[x_8,x_{20}]/(x_8^3,x_{20}^3)\otimes\Lambda(x_3,x_7,x_{15},x_{19},x_{27},x_{35},x_{39},x_{47})$$
$$\otimes\mathbf{Z}/3\{a_4,a_8,a_9,a_{20},a_{21},c_{17},c_{41},b_{16},b_{40},d_{28},e_{36},e_{48}\}/I,$$

$$d(x_{15})=x_8\otimes a_8+x_7\otimes a_9,\quad d(x_{39})=x_{20}\otimes a_{20}+x_{19}\otimes a_{21},$$
$$d(x_{27})=x_8\otimes a_{20}+x_{19}\otimes a_9+x_{20}\otimes a_8+x_7\otimes a_{21},$$

$$
\begin{aligned}
d(x_{35}) &= x_8 \otimes d_{28} + x_{27} \otimes a_9 - x_8^2 \otimes a_{20} + x_{19} \otimes c_{17} + x_{20} \otimes b_{16} \\
&\quad + x_{15} \otimes a_{21} + x_{20}x_8 \otimes a_8, \\
d(x_{47}) &= x_8 \otimes b_{40} + x_{39} \otimes a_8 + x_{20} \otimes d_{28} + x_{27} \otimes a_{21} + x_7 \otimes c_{41} \\
&\quad - x_{20}^2 \otimes a_8 + x_{20}x_8 \otimes a_{20},
\end{aligned}
$$

$$
d|_{\mathbf{Z}/3\{\ \}/I} = \text{the ordinary differential on } \mathbf{Z}/3\{\ \}/I,
$$

$$
\begin{aligned}
a_9 \cdot x_{15} &= -x_{15} \otimes a_9 + x_7 \otimes c_{17}, & a_{21} \cdot x_{39} &= -x_{39} \otimes a_{21} + x_{19} \otimes c_{41}, \\
a_9 \cdot x_{27} &= -x_{27} \otimes a_9 + x_{19} \otimes c_{17}, & a_{21} \cdot x_{27} &= -x_{27} \otimes a_{21} + x_7 \otimes c_{41}, \\
a_9 \cdot x_{35} &= -x_{35} \otimes a_9 + x_{27} \otimes c_{17}, & a_{21} \cdot x_{35} &= -x_{35} \otimes a_{21} + x_{15} \otimes c_{41}, \\
a_9 \cdot x_{47} &= -x_{47} \otimes a_9 + x_{39} \otimes c_{17}, & a_{21} \cdot x_{47} &= -x_{47} \otimes a_{21} + x_{27} \otimes c_{41}.
\end{aligned}
$$

The differential operator d and the bracket [,] are trivial on the generators if they are not indicated above.

References

[1] E. H. Brown, Jr., Twisted tensor products, I, Ann. Math., **69**(1959), 223-246.

[2] E. M. Friedlander, Computations of K-theories of finite fields, Topology, **15**(1976), 87-109.

[3] S. Kleinerman, The cohomology of Chevalley groups of exceptional Lie type, Memoirs of AMS, **268**(1982).

[4] A. Kono, M. Mimura and N. Shimada, Cohomology of classifying space of certain associative H-spaces, J. Math. Kyoto Univ., **15**(1975), 607-617.

[5] A. Kono, M. Mimura and N. Shimada, On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7, J. Pure and Applied Algebra **8**(1977), 755-776.

[6] K. Kono and K. Kozima, The adjoint action of a Lie group on the space on loops, J. Math. Soc. Japan, **45**(1993), 495-510.

[7] K. Kuribayashi, M. Mimura and M. Tezuka, On the cohomology of finite Chevalley groups, (In preparation).

[8] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, Ann. of Math., **81**(1965), 211-236.

[9] M. Mimura, The characteristic classes for the exceptional Lie groups, LMS Lecture Note Series, **175**(1992), 103-130.

[10] M. Mimura and Y. Sambe, On the cohomology mod p of the classifying spaces of the exceptional Lie groups, I, J. Math. Kyoto Univ., **19**(1979), 553-581.

[11] M. Mimura and Y. Sambe, On the cohomology mod p of the classifying spaces of the exceptional Lie groups, II, J. Math. Kyoto Univ., **20**(1980), 327-349.

[12] M. Mimura and Y. Sambe, Collapsing of Eilenberg-Moore spectral sequence mof 5 of the compact exceptional group E_8, J. Math. Kyoto Univ., **21**(1981), 203-230.

[13] D.C.Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press,(1986).

[14] N. Shimada and A. Iwai, On the cohomology of some Hopf algebras, Nagoya Math. J., **30**(1967), 103-111.

[15] M. Tezuka, On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces, Cohomology of Finite Groups and Related Topics(Japanese), KIMS Kokyuroku, **1057**(1998), 54-55.